

# NON-NILPOTENT SUBGROUPS OF LOCALLY GRADED GROUPS

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**ABSTRACT.** In this paper, we show that a locally graded group with a finite number  $m$  of non-(nilpotent of class at most  $n$ ) subgroups is (soluble of class at most  $\lfloor \log_2(n) \rfloor + m + 3$ )-by-(finite of order  $\leq m!$ ). Also we show that the derived length of a soluble group with a finite number  $m$  of non-(nilpotent of class at most  $n$ ) subgroups, is at most  $\lfloor \log_2(n) \rfloor + m + 1$ .

**Keywords.** norm, Schmidt group, derived length, locally graded group.

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## 1. Introduction and results

Let  $G$  be a group. A non-nilpotent finite group whose proper subgroups are all nilpotent is well-known (called Schmidt group). In 1924, O.Yu. Schmidt studied such groups and proved that such groups are soluble [7]. Subsequently, Newman and Wiegold in [5], discussed infinite non-nilpotent groups whose proper subgroups are all nilpotent. Such groups need not be soluble in general. For example, the *Tarski Monsters*, which are infinite simple groups with all proper subgroups of a fixed prime order.

Following [9] we say that a group  $G$  is a  $\mathcal{S}^m$ -group if  $G$  has exactly  $m$  non-nilpotent subgroups. More recently Zarrin in [9] generalized Schmidt's Theorem and proved that every finite  $\mathcal{S}^m$ -group with  $m < 22$  is soluble. Let  $n$  be a non-negative integer. We say that a group  $G$  is an  $\mathcal{S}_n^m$ -group, if  $G$  has exactly  $m$  non-(nilpotent of class at most  $n$ ) subgroups. Clearly, every  $\mathcal{S}_n^m$ -group is a  $\mathcal{S}^r$ -group, for some  $r \leq m$ . Here, we show that every locally graded group with a finite number  $m$  of non-(nilpotent of class at most  $n$ ) subgroups, is soluble-by-finite. Recall that a group  $G$  is locally graded if every non-trivial finitely generated subgroup of  $G$  has a non-trivial finite homomorphic image. This is a rather large class of groups, containing for instance all residually finite groups and all locally(soluble-by-finite) groups.

**Theorem A.** Every locally graded  $\mathcal{S}_n^m$ -group is (soluble of class at most  $\lfloor \log_2(n) \rfloor + m + 3$ )-by-(finite of order  $\leq m!$ ).

This result suggests that the behavior of non-(nilpotent of class at most  $n$ ) subgroups has a strong influence on the structure of the group.

Finding a upper bound for the solubility length of a soluble group is an important problem in the theory of groups, for example see [8]. It is well-known that a nilpotent group of class  $n$  (or a group without non-(nilpotent of class at most  $n$ ) subgroups) has derived length  $\leq \lfloor \log_2(n) \rfloor + 1$  (see [6], Theorem 5.1.12). Here,

we obtain a result which is of independent interest, namely, the derived length of soluble  $\mathcal{S}_n^m$ -groups is bounded in terms of  $m$  and  $n$ . (Note that every nilpotent group of class  $n$  is a  $\mathcal{S}_n^m$ -group with  $m = 0$ .)

**Theorem B.** Let  $G$  be a soluble  $\mathcal{S}_n^m$ -group and  $d$  be the derived length of  $G$ . Then  $d \leq \lfloor \log_2(n) \rfloor + m + 1$ .

## 2. Proofs

If  $G$  is an arbitrary group, the *norm*  $B_1(G)$  of  $G$  is the intersection of the normalizers of all subgroups of  $G$  and  $W(G)$  is the intersection of the normalizers of all subnormal subgroups of  $G$ . In 1934 and 1958, respectively, those concepts were considered by R. Baer and Wielandt (see also [1, 2, 3]). More recently Zarrin generalized this concept in [10]. Here we define  $A_n(G)$  as the intersection of all the normalizers of non-(nilpotent of class at most  $n$ ) subgroups of  $G$ , i.e.,

$$A_n(G) = \bigcap_{H \in \mathfrak{T}_n(G)} N_G(H),$$

where  $\mathfrak{T}_n(G) = \{H \mid H \text{ is a non-(nilpotent of class at most } n) \text{ subgroup of } G\}$  (with the stipulation that  $A_n(G) = G$  if all subgroups of  $G$  are nilpotent of class at most  $n$ ). Clearly

$$B_1(G) \leq A_i(G) \leq A_{i+1}(G).$$

Moreover, in view of the proof of Theorem A, below, we can see that, for every locally graded group  $G$ , we have

$$A_n(G) \text{ is a soluble normal subgroup of } G \text{ of class } \leq \lfloor \log_2(n) \rfloor + 4.$$

**Proof of Theorem A.** The group  $G$  acts on the set  $\mathfrak{T}_n(G)$  by conjugation. By assumption  $|\mathfrak{T}_n(G)| = m$ . It is easy to see that the subgroup  $A_n(G)$  is the kernel of this action and so  $A_n(G)$  is normal in  $G$  and  $G/A_n(G)$  is embedded in  $S_m$ , the symmetric group of degree  $m$ . So

$$|G/A_n(G)| \leq m!.$$

Therefore to complete the proof it is enough to show that  $H = A_n(G)$  is soluble of class at most  $\lfloor \log_2(n) \rfloor + 4$ . To see this, it is enough to show that  $K = H^{(3)}$  is nilpotent of class at most  $n$ . Suppose on the contrary that  $K$  is not nilpotent of class at most  $n$ . It follows that every subgroup containing  $K$  is not nilpotent of class at most  $n$  and so, by definition of  $A_n(G)$ , it is a normal subgroup of  $H$ . Therefore every subgroup of  $H/K$  is normal. That is,  $H/K$  is a Dedekind group and so, it is well-known (see [6], Theorem 5.3.7), that  $H/K$  is metabelian. From which it follows that

$$H^{(2)} = H^{(3)} = K. \quad (\bullet)$$

We claim the following conclusions.

**Step1.** Every proper normal subgroup of  $K$  is nilpotent of class at most  $n$ .

Suppose, a contrary, that there exists a proper normal subgroup  $M$  of  $K = H^{(2)}$  such that  $M$  is not nilpotent of class at most  $n$ . Then we can obtain, by definition of  $A_n(G)$ , that  $H^{(2)}/M$  is a Dedekind group (so it is metabelian) and so, in view of  $(\bullet)$ ,  $H^{(2)} = M$ , a contradiction.

**Step2.** The product of all proper normal subgroups of  $K$ , say  $R$ , is a proper nilpotent subgroup of  $K$  of class at most  $n$ .

Suppose that  $M_1, M_2, \dots, M_t$  are proper normal subgroups of  $H^{(2)}$ . Then, by step 1, every  $M_i$  is soluble and so  $M_1 M_2 \dots M_t$  is soluble. Now by  $(\bullet)$ , we conclude that  $H^{(2)} \neq M_1 M_2 \dots M_t$ . Therefore  $M_1 M_2 \dots M_t$  is a proper normal subgroup of  $H^{(2)}$  and so, by step 1, it is nilpotent of class at most  $n$ . Therefore  $R$  is a locally nilpotent of class at most  $n$  group, and so  $R$  is nilpotent of class at most  $n$  (note that the class of nilpotent groups of class at most  $n$  is locally closed). Also as  $(\bullet)$ , we have  $R \neq H^{(2)}$ .

**Step3.** Finishing the proof.

We note that, by definition of  $A_n(G)$ , every subgroup of  $H^{(2)}$  which is not nilpotent of class at most  $c$  is a normal subgroup of  $H^{(2)}$ . It follows, as  $H^{(2)}/R$  is a simple group, that all proper subgroups of  $H^{(2)}/R$  are nilpotent of class at most  $n$ . Since  $H^{(2)}$  is locally graded, by the main result of [4],  $H^{(2)}/R$  is locally graded. Therefore if  $H^{(2)}/R$  is finitely generated then it must be finite. Thus, by Schmidt's Theorem,  $H^{(2)}/R$  is soluble, which is contrary to  $(\bullet)$ . If  $H^{(2)}/R$  is not finitely generated, then  $H^{(2)}/R$  is locally nilpotent of class at most  $n$  and so  $H^{(2)}/R$  is nilpotent of class at most  $n$ , a contradiction.

Now we prove Theorem B.

**Proof of Theorem B.** Assume that a soluble group  $G$  has derived length  $> [\log_2 n] + 1 + m$  for some  $n, m \geq 1$ . Then obviously the  $m + 1$  derived subgroups  $G, G', \dots, G^{(m)}$  are all pairwise distinct and have solubility length  $> [\log_2 n] + 1$ . Therefore they cannot be nilpotent of class at most  $n$ . This shows that  $G$  cannot be a  $S_n^m$ -group, a contradiction.

Finally, as every  $S_n^m$ -group is a  $S^r$ -group, for some  $r \leq m$ , and by the main result in [9], we can see that every  $S_n^m$ -group with  $m \leq 21$  is soluble. Hence the following question arises naturally:

**Question 2.1.** *Assume that  $G$  is a  $S_n^m$ -group. What relations between  $m, n$  guarantee that  $G$  is soluble?*

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